COURSE CODE: MTH 203

COURSE TITLE: Sets, Logic and Algebra I

COURSE UNITS: 2 Units

MODULE 1

Lesson 1: Introduction to Set Theory

Introduction

Set theory is a fundamental branch of mathematical logic that studies sets, which are collections of objects. While seemingly abstract, the concepts of set theory provide a powerful language and framework for understanding and organizing information across various disciplines, including computer science, mathematics, statistics, and even areas like linguistics and philosophy. In computer science, set theory is foundational for topics such as database design, algorithm analysis, formal languages, and the very logic underlying operating systems. This initial lesson will introduce the basic **definitions and types of sets**, laying the groundwork for understanding how collections of objects can be formally described and manipulated. We will then explore fundamental **set operations** such as union, intersection, and complement, which allow us to combine and compare sets in meaningful ways.

Lesson Outcomes

Upon completion of this lesson, you will be able to:

- **Define** the concept of a set and its elements.
- **Distinguish** between different types of sets, including finite, infinite, empty, singleton, subsets, proper subsets, universal sets, and power sets.
- **Represent** sets using different notations (roster method, set-builder notation).
- **Perform** basic set operations: union, intersection, and complement.
- **Explain** and apply the properties of these set operations.
- **Use Venn diagrams** to visually represent sets and set operations.
- **Solve** simple problems involving sets and set operations.

1. Definitions and Types of Sets

Definition of a Set:

A **set** is a well-defined collection of distinct objects, considered as an entity in itself. The objects that make up a set are called its **elements** or **members**.

- **Well-defined:** This means that for any given object, it is clear whether that object belongs to the set or not. There should be no ambiguity. For example, "the set of tall people" is not well-defined because "tall" is subjective. However, "the set of people in this room whose height is greater than 1.8 meters" is well-defined.
- **Distinct:** Each element in a set must be unique. A set does not contain duplicate elements. For example, the collection $\{1, 2, 2, 3\}$ is not a set in the formal sense; the set would be $\{1, 2, 3\}$.
- **Collection of objects:** The objects can be anything: numbers, letters, people, other sets, etc.

Representing Sets:

There are two common ways to represent sets:

- **Roster Method (Tabular Form):** Listing all the elements of the set within curly braces {} and separating them by commas. The order of elements does not matter.
  - Example 1: The set of the first three positive integers can be written as

$$A = \{1,2,3\}.$$

  - Example 2: The set of vowels in the English alphabet can be written as

$$V = \{a, e, i, o, u\}.$$

- **Set-Builder Notation (Descriptive Form):** Defining the set by specifying a property or condition that its elements must satisfy. The general form is {x|P(x)}, which reads as "the set of all x such that x has the property P".
  - Example 1: The set of even positive integers can be written as

$$E = \{x \mid x \text{ is a positive integer and } x \text{ is divisible by } 2\}.$$

- Example 2: The set of all students enrolled in a particular course can be written as $S = \{s \mid s\ is\ a\ student\ enrolled\ in\ CSC101\}$.

Types of Sets:

- **Finite Set:** A set that contains a finite number of elements. The number of elements in a finite set is called its cardinality, denoted by $\mid A \mid$ or $n(A)$.
  - Example: $A = \{1,2,3,4,5\}, \mid A \mid = 5$.
- **Infinite Set:** A set that contains an infinite number of elements. It is impossible to list all its elements.
  - Example: The set of natural numbers $N = \{1,2,3, \ldots\}$.
- **Empty Set (Null Set):** A set that contains no elements. It is denoted by $\emptyset\ or\ \{\}$. The cardinality of the empty set is $0, \mid \emptyset \mid = 0$. The empty set is a subset of every set.
- **Singleton Set:** A set that contains exactly one element.
  - Example: $B = \{7\}, \mid B \mid = 1$.
- **Subset:** A set A is a **subset** of a set B, denoted by $A \subseteq B$, if every element of A is also an element of B.
  - Example: If $A = \{1,2\}$ and $B = \{1,2,3\}$, then $A \subseteq B$.
  - Note: Every set is a subset of itself (A⊆A), and the empty set is a subset of every set ($\emptyset \subseteq B$).
- **Proper Subset:** A set A is a **proper subset** of a set B, denoted by $A \subset B$ (or sometimes $A \subsetneq B), if\ A \subseteq B\ and\ A\ \ = B$. This means that every element of A is in B, and there is at least one element in B that is not in A.
  - Example: If $A = \{1,2\}\ and\ B = \{1,2,3\}, then\ A \subset B$.
- **Universal Set:** A set that contains all the elements under consideration in a particular context. It is usually denoted by U or U. The universal set defines the boundary of the elements we are interested in.
  - Example: If we are discussing sets of digits, the universal set might be $U = \{0,1,2,3,4,5,6,7,8,9\}$.
- **Power Set:** The **power set** of a set A, denoted by P(A) or 2A, is the set of all possible subsets of A, including the empty set and A itself. If a set A has n elements, then its power set P(A) has 2n elements.
  - Example: If A={a,b}, then the subsets of A are ∅,{a},{b},{a,b}. Therefore, the power set of A is P(A)={∅,{a},{b},{a,b}}, and |P(A)|=22=4.

Equality of Sets:

Two sets A and B are said to be **equal**, denoted by A=B, if they contain exactly the same elements, regardless of the order in which the elements are listed. This is formally defined as $A = B$ if and only if $A \subseteq B\ and\ B \subseteq A$. * Example: $\{1,2,3\} = \{3,1,2\}$.

2. Set Operations (Union, Intersection, Complement)

Now, let's explore the fundamental operations that can be performed on sets to create new sets.

Union (A∪B):

The **union** of two sets A and B, denoted by A∪B, is the set containing all elements that are in A, or in B, or in both. * In set-builder notation: $A \cup B = \{x \mid x \in A\ or\ x \in B\}$. * Example: If $A = \{1,2,3\}\ and\ B = \{3,4,5\}, then\ A \cup B = \{1,2,3,4,5\}$. Note that the common element 3 appears only once in the union.

Intersection (A∩B):

The **intersection** of two sets A and B, denoted by $A \cap B$, is the set containing all elements that are common to both A and B. * In set-builder notation: $A \cap B = \{x \mid x \in A\ and\ x \in B\}$. * Example: If $A = \{1,2,3\}\ and\ B = \{3,4,5\}, then\ A \cap B = \{3\}$.

Disjoint Sets:

Two sets A and B are said to be **disjoint** if their intersection is the empty set, i.e., $A \cap B = \emptyset$. This means they have no elements in common. * Example: If $A = \{1,2\}\ and\ B = \{3,4\}$, then $A \cap B = \emptyset$, so A and B are disjoint.

Complement ($A'\ or\ Ac\ or\ A^{-}$):

The **complement** of a set A, denoted by A′ (or Ac or A¯), is the set of all elements in the universal set U that are not in A. The complement is always defined with respect to a universal set. * In set-builder notation: $A' = \{x \mid x \in U\ and\ x \in/A\}$. * Example: If the universal set $U = \{1,2,3,4,5\}$ and $A = \{1,3\}$, then $A' = \{2,4,5\}$.

Venn Diagrams:

Venn diagrams are visual representations of sets and their relationships using overlapping circles or other closed curves within a rectangle that represents the universal set.

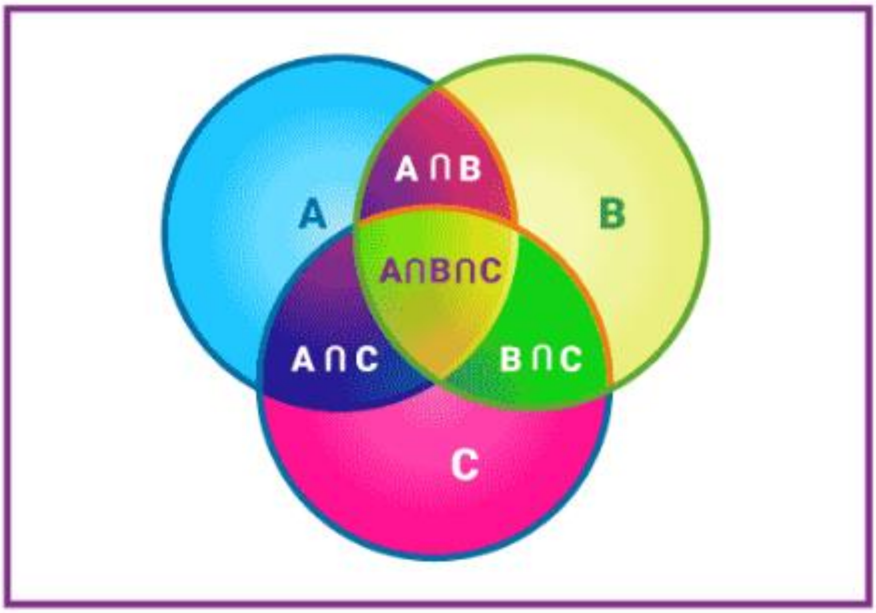

Figure 1: Venn Diagram (https://www.quora.com/Why-cant-we-draw-a-Venn-diagram-for-4-sets-with-circles-and-how-can-we-solve-it)

- A single set A is represented by a circle. The area inside the circle represents the elements of A, and the area outside represents the elements not in A (but within U).
- The union A∪B is represented by the shaded area that covers both circles.
- The intersection A∩B is represented by the shaded area where the two circles overlap.
- The complement A′ is represented by the shaded area outside the circle representing A but inside the rectangle representing U.

Venn diagrams are a useful tool for visualizing set operations and understanding relationships between sets.

Properties of Set Operations:

The basic set operations have several important properties:

- **Commutative Laws:**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$

- **Associative Laws:**
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- **Distributive Laws:**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **Identity Laws:**
  - $A \cup \emptyset \lt /0 \gt = A$
  - $A \cap U = A$
  - $A \cup U = U$
  - $A \cap \emptyset = \emptyset$
- **Complement Laws:**
  - $A \cup A' = U$
  - $A \cap A' = \emptyset$
  - $(A')' = A\ (Double\ complement)$
- **De Morgan's Laws:**
  - $(A \cup B)' = A' \cap B'$
  - $(A \cap B)' = A' \cup B'$

These properties are fundamental and can be used to simplify expressions involving set operations and to prove other results in set theory.

Solving Simple Problems:

Let $U = \{1,2,3,4,5,6,7,8,9,10\}, A = \{1,3,5,7,9\}, and\ B = \{2,3,5,7\}$. Find:

- $A \cup B = \{1,2,3,5,7,9\}$
- $A \cap B = \{3,5,7\}$
- $A' = \{2,4,6,8,10\}$
- $B' = \{1,4,6,8,9,10\}$
- $(A \cup B)' = \{4,6,8,10\}$
- $A' \cap B' = \{2,4,6,8,10\} \cap \{1,4,6,8,9,10\} = \{4,6,8,10\}$ (Verifying De Morgan's Law)

Set Theory:

Understanding set theory provides a foundation for logical reasoning and problem-solving, skills that are valuable in any field, including the growing technology sector Concepts like well-defined collections and relationships between them are applicable in areas such as organizing data, designing algorithms, and understanding the fundamentals of how computers process information. As digital literacy and technological advancements continue in the region, a grasp of these basic mathematical principles can be a significant asset.

Summary

In this lesson, we introduced the fundamental concepts of set theory. We defined a set as a well-defined collection of distinct objects and explored different types of sets, including finite, infinite, empty, singleton, subsets, proper subsets, universal sets, and power sets. We learned how to represent sets using the roster method and set-builder notation. We then focused on the basic set operations: union, intersection, and complement, understanding their definitions, properties (commutative, associative, distributive, identity, complement, and De Morgan's laws), and how to visualize them using Venn diagrams. We also solved simple problems involving these set operations. This foundational knowledge of set theory is crucial for further studies in mathematics, computer science, and related fields.

Evaluation Questions

1. Define a set and explain the terms "well-defined" and "distinct" in this context. Give an example of a collection that is not a set according to the formal definition.
2. Differentiate between the roster method and set-builder notation for representing sets. Provide an example of a set represented in each way.
3. Explain the difference between a subset and a proper subset. If set $A = \{a, b, c\}$, list all its subsets and identify which ones are proper subsets of A.
4. Define the union, intersection, and complement of two sets A and B. Use set-builder notation to express each of these operations.
5. State De Morgan's Laws for set operations. Use a Venn diagram to illustrate one of these laws.

Suggested Answers

1. A **set** is a well-defined collection of distinct objects. "Well-defined" means it is clear whether an object belongs to the collection or not. "Distinct" means each object in the collection is unique. Example of a non-set: "the set of interesting books" (interesting is subjective).
2. The **roster method** lists all elements within curly braces, e.g., $A = \{1,2,3\}$. Set-builder notation defines a set by aproperty e.g,

$$B = \{x \mid x \; is \; an \; even \; positive \; integer \; less \; than \; 10\}.$$

3. A set A is a **subset** of B $(A \subseteq B)$ if every element of A is in B. A set A is a **proper subset** of $B\,(A \subset B)$ if $A \subseteq B$ and $A \quad = B$. Subsets of $A = \{a, b, c\}$ are: $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. Proper subsets are all of these except $\{a, b, c\}$.

4.
   - **Union:** $A \cup B = \{x \mid x \in A \; or \; x \in B\}$
   - **Intersection:** $A \cap B = \{x \mid x \in A \; and \; x \in B\}$
   - **Complement:** $A' = \{x \mid x \in U \; and \; x \notin A\}$ (where U is the universal set)

5. De Morgan's Laws:
   - $(A \cup B)' = A' \cap B'$
   - $(A \cap B)' = A' \cup B'$ A Venn diagram for $(A \cup B)' = A' \cap B'$ would show two overlapping circles (A and B) within a rectangle (U). The union A∪B is the area covered by both circles. The complement $(A \cup B)'$ is the area outside both circles. A' is the area outside circle A, and B′ is the area outside circle B. The intersection $A' \cap B'$ is the area that is outside both A and outside B, which is the same as $(A \cup B)'$. This area would be shaded to illustrate the law.